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Different Interpretations of the Particle Production in Quantum Fields Theory

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Abstract We discuss different interpretations of the particle production in quantum fields theory. We study in detail the "instantaneous Hamiltonian diagonalization method" and the "adiabatic vacuum prescription" and we show the difference between both methods calculating the particle production in the flat Friedmann-Robertson-Walker chart of the de Sitter space-time. Finally, we study the particle production in a classically forbidden region, and we apply the obtained results to the problem of the catastrophic particle production in tunneling universes.

Keywords Vacuum state · Particle production · Euclidean quantum fields theory

1 Introduction

Several definitions of the vacuum state exist in quantum fields theory that, when one does not calculate "in-out" transitions, can give rise to different results when one calculates the number of produced particles. The aim of the present paper is to discuss in detail the different interpretations, that exist in the literature, of the particle production phenomenon, specially the "instantaneous Hamiltonian diagonalization method" [1, 2] and the "adiabatic vacuum prescription" [3–6]. To do this, we review the definitions of these methods and we deduce, in Heisenberg and Schrödinger pictures, the different computational formulae that give rise both prescriptions, and we also compare these two interpretations with the interpretation used in [7–10].

As an application we study two different examples: Firstly we examine the particle production at late times from the Bunch-Davies [11] vacuum state in the Friedmann-Robertson-Walker flat chart of the de Sitter space-time, where the adiabatic vacuum prescription give the thermal spectrum obtained by Gibbons and Hawking [12, 13], and the diagonalization method give a greater particle production [14]. We will see that this difference is due to the

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fact that does not exist a well defined "out" region at late times. In the second example we study the particle production by a constant electric field, the number of produced particles is given by the well-known Schwinger's formula [15], independently of the method used to calculate it.

We are also interested in the particle creation in an Euclidean region [16]. In this case, we must use the Schrödinger picture because the evolution operator is not unitary and then one cannot use the Heisenberg picture to calculate matrix elements. Then studying the Schrödinger equation we will see that, for some initial quantum states, there is a catastrophic particle production, and we will show what kind of initial conditions avoid this phenomenon. Finally, we apply these results to the problem of the catastrophic particle production in a minisuperspace model where a closed homogeneous universe, filled with a uniform massive scalar field conformally coupled to gravity, is created from nothing [17–19].

2 Heisenberg Picture in a Lorentzian Region

We consider the Hamiltonian operator $\hat{\mathcal{H}}(\eta) \equiv \frac{1}{2}(\hat{\sqcap}^2 + \omega^2(\eta)\hat{\phi}^2) - \frac{1}{2}\hbar\omega(\eta)$. In the Heisenberg picture the operators $\hat{\sqcap}_H$ and $\hat{\phi}_H$ satisfy, in a Lorentzian region, the equations $\hat{\phi}'_H = \hat{\sqcap}_H$ and $\hat{\sqcap}'_H = -\omega^2\hat{\phi}_H$, that is, $\hat{\phi}_H$ satisfy the Klein-Gordon equation $\hat{\phi}''_H + \omega^2\hat{\phi}_H = 0$, and thus, we can write:

$$\begin{pmatrix} \hat{\phi}_H(\eta) \\ \hat{\sqcap}_H(\eta) \end{pmatrix} = \begin{pmatrix} \nu(\eta) \\ \nu'(\eta) \end{pmatrix} \hat{A}_{\nu,H} + \begin{pmatrix} \nu^*(\eta) \\ \nu'^*(\eta) \end{pmatrix} \hat{A}^\dagger_{\nu,H}, \tag{1}$$

where the mode function ν is a solution of the Klein-Gordon equation and $\hat{A}_{\nu,H}$ is a constant operator in the Heisenberg picture that we call the "annihilation operator relative to the mode ν".

From the commutation relation $[\hat{\sqcap}, \hat{\phi}] = -i\hbar$, one can deduce that ν must satisfy the relation $\nu'\nu^* - \nu'^*\nu = -i\hbar$, and then the annihilation operator is given by

$$\hat{A}_{\nu,H} = -\frac{i}{\hbar}\left(\nu'^*(\eta)\hat{\phi}_H(\eta) - \nu^*(\eta)\hat{\sqcap}_H(\eta)\right). \tag{2}$$

We can define the "vacuum state relative to the mode ν", namely $|0_H;\nu\rangle$, as the quantum state that satisfy $\hat{A}_{\nu,H}|0_H;\nu\rangle = 0$. It is clear from this definition that there is not particle production at any time, because $\langle\nu;0_H|\hat{A}^\dagger_{\nu,H}\hat{A}_{\nu,H}|0_H;\nu\rangle = 0$ all the time [3]. However, this definition depends on the choice of the mode ν. Effectively, if we choose two different mode functions, namely ν_1 and ν_2, since we have $\hat{A}_{\nu_1,H} = \alpha_{1,2}\hat{A}_{\nu_2,H} + \beta^*_{1,2}\hat{A}^\dagger_{\nu_2,H}$, with

$$\alpha_{1,2} = -\frac{i}{\hbar}\mathcal{W}[\nu_2;\nu_1^*], \qquad \beta^*_{1,2} = -\frac{i}{\hbar}\mathcal{W}[\nu_2^*;\nu_1^*], \quad \text{where } \mathcal{W} \text{ denotes the Wronskian.} \tag{3}$$

Then an observer in the $|0_H;\nu_2\rangle$ vacuum state, can observes ν_1-particles, because one has $\mathcal{N}_{1,2} \equiv \langle\nu_2;0_H|\hat{A}^\dagger_{\nu_1,H}\hat{A}_{\nu_1,H}|0_H;\nu_2\rangle = |\beta_{1,2}|^2$.

In this way, if we consider the family of solutions to the Klein-Gordon equation, namely $\nu_t(\eta)$, defined by the initial condition

$$\nu_t(t) \equiv f(t); \qquad \nu'_t(t) \equiv g(t), \quad \text{where } f \text{ and } g \text{ are some arbitrary functions,} \tag{4}$$

then the number of ν_τ-particles detected by an observer in the $|0_H;\nu_{\tau'}\rangle$ vacuum state is

$$\mathcal{N}(\tau;\tau') \equiv \langle \nu_{\tau'}; 0_H | \hat{A}^\dagger_{\nu_\tau,H} \hat{A}_{\nu_\tau,H} | 0_H; \nu_{\tau'}\rangle = |\beta(\tau;\tau')|^2, \quad \text{with } \beta(\tau;\tau') = \frac{i}{\hbar}\mathcal{W}[\nu_{\tau'};\nu_\tau]. \tag{5}$$

Note that, different families of solutions give rise to different definitions of the vacuum state. For example, to define the **adiabatic vacuum prescription** first we consider the ϵ-Klein-Gordon equation $\epsilon\nu'' + \omega^2(\eta)\nu = 0$, where here, ϵ is a dimensionless parameter that one shall set $\epsilon = 1$ at the end of the calculations. At order "n", a WKB solution of the Klein-Gordon equation is (see for details [6]):

$$\nu_{n;WKB}(\tau;\epsilon) \equiv \sqrt{\frac{\hbar}{2W_n(\tau;\epsilon)}} e^{-\frac{i}{\epsilon}\int^\tau W_n(\eta;\epsilon)d\eta}, \tag{6}$$

where $W_0 = \omega$ and

$$W_n = \text{terms until order } \epsilon^{2n} \text{ of } \left(\sqrt{w^2 - \epsilon^2\left[\frac{1}{2}\frac{W''_{n-1}}{W_{n-1}} - \frac{3}{4}\frac{(W'_{n-1})^2}{W^2_{n-1}}\right]}\right). \tag{7}$$

Once one has introduced the WKB solutions, the adiabatic vacuum at order n is defined through the family $\nu_{n;t}(\eta)$ that satisfy the initial condition

$$\nu_{n;t}(t) = \nu_{n;WKB}(t;\epsilon=1); \qquad \nu'_{n;t}(t) = \nu'_{n;WKB}(t;\epsilon=1). \tag{8}$$

From this definition, at the n order, the β-Bogoliubov coefficient is $\beta_n(\tau;\tau') = \frac{i}{\hbar}\mathcal{W}[\nu_{n;\tau'};\nu_{n;\tau}]$, and the number of produced particles at time τ can be calculated from the formula

$$\begin{aligned}\mathcal{N}_n(\tau;\tau') \equiv |\beta_n(\tau;\tau')|^2 = {} & \frac{1}{\hbar W_n(\tau)}\left(\frac{1}{2}[|\nu'_{n;\tau'}(\tau)|^2 + W_n^2(\tau)|\nu_{n;\tau'}(\tau)|^2] - \frac{1}{2}\hbar W_n(\tau)\right) \\ & + \frac{W_n'^2(\tau)}{8\hbar W_n^3(\tau)}|\nu_{n;\tau'}(\tau)|^2 + \frac{W_n'(\tau)}{4\hbar W_n^2(\tau)}(\nu'^*_{n;\tau'}(\tau)\nu_{n;\tau'}(\tau) \\ & + \nu'_{n;\tau'}(\tau)\nu^*_{n;\tau'}(\tau)),\end{aligned} \tag{9}$$

where $W_n(\tau) \equiv W_n(\tau;\epsilon=1)$.

Another important family of solutions to the Klein-Gordon equation, namely $\nu_{diag;t}(\eta)$, is given by the initial condition

$$\nu_{diag;t}(t) = \nu_{0;WKB}(t;\epsilon=1); \qquad \nu'_{diag;t}(t) = -i\omega(t)\nu_{0;WKB}(t;\epsilon=1). \tag{10}$$

This family defines the so-called **instantaneous Hamiltonian diagonalization method**, and it's not difficult to prove that the instantaneous mean occupation number at time τ is given by the formula

$$\begin{aligned}\mathcal{N}_{diag}(\tau;\tau') &\equiv |\beta_{diag}(\tau;\tau')|^2 \\ &= \frac{1}{\hbar\omega(\tau)}\left(\frac{1}{2}[|\nu'_{diag;\tau'}(\tau)|^2 + \omega^2(\tau)|\nu_{diag;\tau'}(\tau)|^2] - \frac{1}{2}\hbar\omega(\tau)\right),\end{aligned} \tag{11}$$

which shows that this quantity is the energy at time τ of the mode $\nu_{diag;\tau'}$ divided by the energy, at time τ, of a single particle.

2.1 Examples

In this Section we study two no-trivial examples where we can see the difference between both methods.

Example 2.1 (Particle creation in the flat FRW chart of the de Sitter space-time [13, 14])

We consider the case $\omega(\eta)=\sqrt{\omega_0^2+\frac{A^2}{\eta^2}}$ with $A\gg 1$ and $-\infty<\eta<0$. An easy calculation yields

$$\nu_{0;WKB}(\eta;\epsilon=1)=\sqrt{\frac{\hbar}{2\omega(\eta)}}e^{i\omega_0\sqrt{\eta^2+A^2/\omega_0^2}}\left(\frac{|\eta|}{\sqrt{\eta^2+A^2/\omega_0^2}+A/\omega_0}\right)^{iA}. \tag{12}$$

And clearly, when $\eta\to-\infty$, we have

$$\nu_{0;WKB}(\eta;\epsilon=1)\to\sqrt{\frac{\hbar}{2\omega_0}}e^{-i\omega_0\eta};\qquad \nu'_{0;WKB}(\eta;\epsilon=1)\to -i\omega_0\nu_{0;WKB}(\eta;\epsilon=1). \tag{13}$$

Then, the mode solution that satisfy the initial condition (13) is given in terms of the Hankel functions [11]

$$\nu(\eta)=C\sqrt{\frac{\pi\hbar\eta}{4}}H_\mu^{(2)}(\omega_0\eta), \tag{14}$$

with $\mu\equiv\sqrt{\frac{1}{4}-A^2}\cong iA$, $C\equiv e^{-i(\frac{\pi\mu}{2}+\frac{\pi}{4})}\cong e^{\pi A/2}e^{-i\frac{\pi}{4}}$ and $\eta=e^{-i\pi}|\eta|$.

Using the asymptotic form of the Hankel functions at late times, i.e., when $\eta\sim 0^-$ [20]

$$\nu(\eta)\cong -C\sqrt{\frac{\hbar\eta}{4\pi}}\frac{1}{A}\left[e^{-\pi A}\Gamma(1-iA)\left(\frac{\omega_0\eta}{2}\right)^{iA}-\Gamma(1+iA)\left(\frac{\omega_0\eta}{2}\right)^{-iA}\right], \tag{15}$$

an easy calculation provides that

$$|\beta_{diag}(0;-\infty)|^2\cong\frac{1}{32\pi A^3}|\Gamma(1-iA)|^2e^{\pi A}. \tag{16}$$

Using at this point [20] $|\Gamma(1+z)|^2=\pi z/\sin(\pi z)$, we conclude that, when $\eta\to 0^-$, the number of produced particles, using the diagonalization method, is given by [14]

$$\mathcal{N}_{diag}(0;-\infty)\cong\frac{1}{16A^2}. \tag{17}$$

However if we use the zero order (the other orders give the same result) adiabatic vacuum prescription, the square of the β-Bogoliubov coefficient is given by

$$|\beta_0(0;-\infty)|^2=\frac{1}{\hbar^2}\left|\nu(0)\nu'_{0;WKB}(0;\epsilon=1)-\nu'(0)\nu_{0;WKB}(0;\epsilon=1)\right|^2. \tag{18}$$

Inserting (12) and (15), in this last formula, we obtain

$$\mathcal{N}_0(0;-\infty)\cong\frac{1}{2\pi A}|\Gamma(1+iA)|^2e^{-\pi A}=\left(e^{2\pi A}-1\right)^{-1}. \tag{19}$$

This is the thermal spectrum obtained in the flat FRW chart of the de Sitter space-time [12, 13].

Remark 2.1 The two methods gives a different result because when $\eta \to 0^-$, we have

$$\nu'_{0,WKB}(\eta;\epsilon=1) = -i\omega(\eta)\nu_{0,WKB}(\eta;\epsilon=1) + \frac{1}{2\eta}\nu_{0,WKB}(\eta;\epsilon=1), \tag{20}$$

this is due to the fact that $\lim_{\eta\to 0^-}\omega(\eta)=\infty$, that is, at late time there is not a well-defined "out" region.

Example 2.2 (Schwinger's formula)

Here we consider the frequency [21]

$$\omega(\eta) = \frac{1}{\hbar^2}\left(c^2(p_3+eE\eta)^2 + c^2 p_\perp^2 + m^2c^4\right); \quad \eta \in (-\infty,\infty). \tag{21}$$

If we make the change $y=\sqrt{\frac{2c}{\hbar eE}}(p_3+eE\eta)$, the Klein-Gordon equation behaves

$$u'' + \tilde{\omega}^2(y)u = 0, \tag{22}$$

where, $\tilde{\omega}(y) \equiv \sqrt{\frac{1}{4}y^2 - A}$ and $A \equiv \frac{-1}{2eEc\hbar}(c^2p_\perp^2 + m^2c^4)$.

In this case, the function $\nu_{0;WKW}(y;\epsilon=1) \equiv \sqrt{\frac{\hbar}{2\tilde{\omega}(y)}}e^{-i\int^y \tilde{\omega}(\eta)d\eta}$, has the asymptotic behavior

$$\nu_{0;WKW}(y;\epsilon=1) = \begin{cases} \hbar^{1/2}e^{iy^2/4}(y^2)^{-1/4-iA/2} & y\to-\infty, \\ \hbar^{1/2}e^{-iy^2/4}(y^2)^{-1/4+iA/2} & y\to\infty. \end{cases} \tag{23}$$

Remark 2.2 Note that $\nu'_{0;WKW}(y;\epsilon=1) = -i\omega(y)\nu_{0;WKW}(y;\epsilon=1)$ when $y\to\pm\infty$, this means that the diagonalization method and zero order adiabatic prescription gives the same value for $\beta(\infty;-\infty)$.

On the other hand, a independent set of solutions of (22) is given by the two following parabolic cylinder functions [20]

$$u_1(y) = \exp\left(-\frac{i}{4}y^2\right) M\left(-\frac{i}{2}A+\frac{1}{4},\frac{1}{2},\frac{i}{2}y^2\right), \tag{24}$$

$$u_2(y) = \frac{1}{\sqrt{2}}\exp\left(-\frac{i}{4}y^2\right) y \exp\left(-\frac{i\pi}{4}\right) M\left(-\frac{i}{2}A+\frac{3}{4},\frac{3}{2},\frac{i}{2}y^2\right), \tag{25}$$

where M is the Kummer's function.

We now define the mode solution

$$\nu(y) \equiv \hbar^{1/2}B^{-1}e^{i\pi/8}e^{-\pi A/4}2^{-1/4-iA/2}\varphi(y), \tag{26}$$

with

$$\varphi(y) \equiv \frac{\Gamma(\frac{1}{4}+\frac{i}{2}A)}{\Gamma(\frac{1}{2})}u_1(y) + \frac{\Gamma(\frac{3}{4}+\frac{i}{2}A)}{\Gamma(\frac{3}{2})}u_2(y), \tag{27}$$

and

$$B\equiv \frac{\Gamma(\frac{1}{4}+\frac{i}{2}A)}{\Gamma(\frac{1}{4}-\frac{i}{2}A)}+i\frac{\Gamma(\frac{3}{4}+\frac{i}{2}A)}{\Gamma(\frac{3}{4}-\frac{i}{2}A)}. \tag{28}$$

Then, from the asymptotic behavior of the Kummer's function [20] we can see that

$$\nu(y)\rightarrow \nu_{0;WKW}(y;\epsilon=1),\quad \text{when } y\rightarrow -\infty, \tag{29}$$

and

$$\nu(y)\rightarrow B^{-1}e^{i\pi/4}2^{1-iA}\nu_{0;WKW}(y;\epsilon=1)+CB^{-1}\nu^*_{0;WKW}(y;\epsilon=1),\quad \text{when } y\rightarrow \infty, \tag{30}$$

with

$$C\equiv \frac{\Gamma(\frac{1}{4}+\frac{i}{2}A)}{\Gamma(\frac{1}{4}-\frac{i}{2}A)}-i\frac{\Gamma(\frac{3}{4}+\frac{i}{2}A)}{\Gamma(\frac{3}{4}-\frac{i}{2}A)}.$$

Then from this last formula we can deduce that, in both cases (diagonalization method and zero order adiabatic vacuum prescription), the square of the β-Bogoliubov coefficient is given by

$$|\beta(\infty;-\infty)|^2=|C/B|^2=e^{2\pi A}=\exp\left(-\frac{\pi}{eEc\hbar}(c^2p_\perp^2+m^2c^4)\right), \tag{31}$$

in agreement with the Schwinger's result [15].

3 Schrödinger Picture

Let $\mathcal{T}(\eta;0)$ be the Schrödinger's evolution operator in a Lorentzian region that satisfy $\mathcal{T}(0;0)=Id$. Both pictures are related through the equations:

$$|0_S(\eta);\nu\rangle=\mathcal{T}(\eta;0)|0_H;\nu\rangle;\qquad \hat{A}_{\nu,S}(\eta)=\mathcal{T}(\eta;0)\hat{A}_{\nu,H}\mathcal{T}(0;\eta). \tag{32}$$

From formula (2), and the relations ($\hat{\phi}_S=\phi$ and $\hat{\sqcap}_S=-i\hbar\partial_\phi$) we have

$$\hat{A}_{\nu,S}(\eta)=-\frac{i}{\hbar}\left(\nu'^*(\eta)\phi+i\hbar\nu^*(\eta)\partial_\phi\right). \tag{33}$$

And since $|0_S(\eta);\nu\rangle$ satisfy the Schrödinger equation and the relation $\hat{A}_{\nu,S}(\eta)|0_S(\eta);\nu\rangle=0$, we easily deduce that the solution of the Schrödinger equation with initial condition $|0_H;\nu\rangle$ is given by the Gaussian state

$$|0_S(\eta);\nu\rangle=\left(\frac{1}{2\pi(\nu^*(\eta))^2}\right)^{1/4}e^{i/2\int_0^\eta\omega(\tau)d\tau}e^{\frac{i}{2\hbar}\frac{\nu'^*(\eta)}{\nu^*(\eta)}\phi^2}. \tag{34}$$

If we use the formula

$$e^{-\frac{R}{2\hbar}\phi^2}=\left(\frac{\pi\hbar}{\omega}\frac{(1+\xi)^2}{1-|\xi|^2}\right)^{1/4}|\xi\rangle, \tag{35}$$

where we have introduced the squeezed state $|\xi\rangle \equiv (1-|\xi|^2)^{1/4} e^{\frac{\xi}{2}(\hat{A}^\dagger)^2}|0\rangle$ with squeezed parameter $\xi \equiv \frac{\omega-R}{\omega+R}$ [22], we can see that the vacuum state relative to the mode ν, i.e. $|0_S(\eta);\nu\rangle$, is a Gaussian squeezed state. In fact we have

$$|0_S(\eta);\nu\rangle = e^{i\alpha(\eta)}|\xi(\eta)\rangle, \quad \text{with } \xi(\eta) = \frac{\nu^*(\eta)\omega(\eta) + i\nu'^*(\eta)}{\nu^*(\eta)\omega(\eta) - i\nu'^*(\eta)}, \tag{36}$$

and $\alpha(\eta)$ is a real function.

Once we have seen that the vacuum state relative to a prescribed mode is a squeezed Gaussian state, we study in detail the instantaneous diagonalization approach in the Schrödinger picture (similar results are obtained for the adiabatic vacuum prescription).

We define the instantaneous annihilation operator at time η, in the Schrödinger picture, as the operator that diagonalize the Hamiltonian at time η, that is

$$\hat{A}(\eta) \equiv \hat{A}_{\nu_{diag;\eta},S}(\eta) = \nu^*_{0;WKB}(\eta)\left(\partial_\phi + \frac{\omega(\eta)}{\hbar}\phi\right). \tag{37}$$

Then the instantaneous vacuum state at time η, namely $|0(\eta)\rangle$, that satisfy $\hat{A}(\eta)|0(\eta)\rangle = 0$ is given by

$$|0(\eta)\rangle \equiv |0_S(\eta);\nu_{diag;\eta}\rangle = \left(\frac{\omega(\eta)}{\pi\hbar}\right)^{1/4} e^{-\frac{\omega(\eta)}{2\hbar}\phi^2}. \tag{38}$$

It's easy to show that

$$\mathcal{N}_{diag}(\tau;\tau') = |\beta_{diag}(\tau;\tau')|^2 = \langle 0(\tau')|\mathcal{T}(\tau',\tau)\hat{A}^\dagger(\tau)\hat{A}(\tau)\mathcal{T}(\tau,\tau')|0(\tau')\rangle, \tag{39}$$

and this is of course, from the point of view of the diagonalization method, the instantaneous number of produced particles at time τ from the vacuum state at time τ', because $\mathcal{T}(\tau,\tau')|0(\tau')\rangle$ is the evolved vacuum state from τ' to τ.

Finally, we are interested in the instantaneous mean occupation number at time τ from the Gaussian squeezed state $|0_S(\tau');\nu\rangle$. An easy calculation yields

$$\begin{aligned}&\langle \nu; 0_S(\tau')|\mathcal{T}(\tau',\tau)\hat{A}^\dagger(\tau)\hat{A}(\tau)\mathcal{T}(\tau,\tau')|0_S(\tau');\nu\rangle\\ &\quad = \frac{1}{\hbar\omega(\tau)}\left(\frac{1}{2}[|\nu(\tau)|^2 + \omega^2(\tau)|\nu(\tau)|^2] - \frac{1}{2}\hbar\omega(\tau)\right)\end{aligned} \tag{40}$$

this means that, in the diagonalization approach, the number of produced particles at time τ from a Gaussian squeezed state is given by the energy of the mode ν at time τ divided by the energy of a particle at that time.

To finish the Section we want to analyze another method to interpret the particle production phenomenon [7–10]. One fixes ν, then the operator $\hat{A}_{\nu,S}(\tau)$ is considered as the annihilation operator at time τ relative to the mode ν, and the state $|0_S(\tau);\nu\rangle$ is considered as the vacuum state at time τ relative to the mode ν. Then, in [7–10] the authors define the number of produced particles at time τ from the vacuum state at time τ', namely $\mathcal{N}_\nu(\tau;\tau')$, in the following way $\mathcal{N}_\nu(\tau;\tau') \equiv \langle\nu;0_S(\tau')|\hat{A}^\dagger_{\nu,S}(\tau)\hat{A}_{\nu,S}(\tau)|0_S(\tau');\nu\rangle$. To calculate $\mathcal{N}_\nu(\tau;\tau')$ we use the following relation

$$\hat{A}_{\nu,S}(\tau) = -\frac{i}{\hbar}\mathcal{W}[\nu(\tau');\nu^*(\tau)]\hat{A}_{\nu,S}(\tau') - \frac{i}{\hbar}\mathcal{W}[\nu^*(\tau');\nu^*(\tau)]\hat{A}^\dagger_{\nu,S}(\tau'), \tag{41}$$

where $\mathcal{W}[f(\tau'); g(\tau)] \equiv f(\tau')g'(\tau) - f'(\tau')g(\tau)$. Then, one obtains [10]

$$\mathcal{N}_\nu(\tau;\tau') = \frac{1}{\hbar^2}\left|\mathcal{W}[\nu(\tau');\nu(\tau)]\right|^2, \tag{42}$$

which is, in general, different from zero.

However we have seen, using the Heisenberg picture, that an observer in the ν vacuum state never observes ν particles. The standard interpretation [3] say us that the number of ν particles produced at time τ, from the ν vacuum state at time τ' is

$$\begin{aligned}\langle \nu; 0_S(\tau')|\mathcal{T}(\tau',\tau)\hat{A}^\dagger_{\nu,S}(\tau)\hat{A}_{\nu,S}(\tau)\mathcal{T}(\tau,\tau')|0_S(\tau');\nu\rangle &= \langle \nu; 0_S(\tau)|\hat{A}^\dagger_{\nu,S}(\tau)\hat{A}_{\nu,S}(\tau)|0_S(\tau);\nu\rangle \\ &= 0. \end{aligned} \tag{43}$$

Moreover, in the particular case $\nu = \nu_{diag;\tau'}$ (note that $|0_S(\tau');\nu_{diag;\tau'}\rangle$ is the instantaneous vacuum state at time τ'), a simple calculation yields

$$\mathcal{N}_{\nu_{diag;\tau'}}(\tau;\tau') = \frac{1}{\hbar\omega(\tau')}\left(\frac{1}{2}[|\nu'_{diag;\tau'}(\tau)|^2 + \omega^2(\tau')|\nu_{diag;\tau'}(\tau)|^2] - \frac{1}{2}\hbar\omega(\tau')\right), \tag{44}$$

which does not agree with the formula (11), and do not has an easy interpretation.

A final remark is in order:

If one has an "in" and an "out" regions, (for example at $-\infty$ and at ∞ respectively), the adiabatic conditions holds ($\omega' \ll \omega^2$), and we choose

$$\nu_{in} \equiv \sqrt{\frac{\hbar}{2\omega_{in}}}e^{-i\omega_{in}\eta} \quad \text{when } \eta \to -\infty, \tag{45}$$

then, using (42) and the zero order WKB approximation, one has $\mathcal{N}_{\nu_{in}}(\infty;-\infty) \cong ((\omega_{out} - \omega_{in})/2)^2$, and thus, this number is different of zero when $\omega_{out} \neq \omega_{in}$, however it's well-known that, in this situation,

$$\mathcal{N}_{diag}(\infty;-\infty) = \mathcal{N}_n(\infty;-\infty) \cong 0. \tag{46}$$

This means, for us, that the interpretation of the particle production used in [7–10] do not give the desired results, and for this reason we believe that is not a good interpretation.

3.1 The Schrödinger Equation in an Euclidean Region

In this Section we study the particle production in an Euclidean region. To do this, we must work in the Schrödinger picture because the evolution operator is not unitary an then on cannot use the Heisenberg picture to calculate matrix elements (see for details [17, 18]).

The problem that we want to study is

$$\begin{cases} \pm\hbar\partial_\eta|\Phi_\pm\rangle = \hat{\mathcal{H}}(\eta)|\Phi_\pm\rangle & \eta_1 \le \eta \le \eta_2 \\ |\Phi_\pm(\eta_{0,\pm})\rangle = (\frac{\omega(\eta_{0,\pm})}{\pi\hbar})^{1/4}e^{-\frac{\omega(\eta_{0,\pm})}{2\hbar}\phi^2} & \eta_{0,\pm} \in [\eta_1,\eta_2], \end{cases} \tag{47}$$

which solution

$$|\Phi_\pm(\eta)\rangle = \left(\frac{1}{2\pi\nu_\pm^2(\eta)}\right)^{1/4} e^{\mp 1/2\int_{\eta_{0,\pm}}^{\eta}\omega(\tau)d\tau} e^{\pm\frac{1}{2\hbar}\frac{\nu'_\pm(\eta)}{\nu_\pm(\eta)}\phi^2}, \tag{48}$$

where $\nu_\pm(\eta)$ is the solution of the problem

$$\nu_\pm'' - \omega^2\nu_\pm = 0; \quad \nu_\pm(\eta_0) = \sqrt{\frac{\hbar}{2\omega(\eta_0)}} \quad \nu_\pm'(\eta_0) = \mp\omega(\eta_0)\nu_\pm(\eta_0). \tag{49}$$

We write this state as a squeezed one

$$|\Phi_\pm(\eta)\rangle = \left(\frac{1}{2\pi\nu_\pm^2(\eta)}\right)^{1/4} e^{\mp 1/2\int_{\eta_{0,\pm}}^{\eta}\omega(\tau)d\tau} \left(\frac{\pi\hbar}{\omega}\frac{1+\xi_\pm(\eta)}{1-\xi_\pm(\eta)}\right)^{1/4} |\xi_\pm(\eta)\rangle, \tag{50}$$

with $\xi_\pm(\eta) = \frac{\omega(\eta)\nu_\pm(\eta)\pm\nu'(\eta)}{\omega(\eta)\nu_\pm(\eta)\mp\nu'(\eta)}$.

Thus, the number of produced particles at time η, from the state $|\Phi_\pm(\eta_{0,\pm})\rangle$ using the diagonalization method, is given now by [16]

$$\mathcal{N}_\pm(\eta) \equiv \frac{\langle\Phi_\pm(\eta)|\hat{A}^\dagger(\eta)\hat{A}(\eta)|\Phi_\pm(\eta)\rangle}{\langle\Phi_\pm(\eta)|\Phi_\pm(\eta)\rangle} = \frac{\xi_\pm^2(\eta)}{1-\xi_\pm^2(\eta)}. \tag{51}$$

This formula is only valid for $|\xi_\pm(\eta)| < 1$, when $|\xi_\pm(\eta)| \geq 1$ the number of produced particles diverges (one has a catastrophic particle production).

To understand this phenomenon note that $\xi_\pm(\eta)$ satisfy the equation

$$\xi_\pm' = \pm 2\omega\xi + \frac{\omega'}{2\omega}(1-\xi_\pm^2), \quad \xi_\pm(\eta_{0,\pm}) = 0. \tag{52}$$

Then, when the adiabatic condition is fulfilled ($\omega' \ll \omega^2$) we can neglect the quadratic term in (52), and an approximate solution is:

$$\xi_\pm(\eta) = e^{\pm 2\int_{\eta_{0,\pm}}^{\eta}\omega d\eta'} \int_{\eta_{0,\pm}}^{\eta} e^{\mp 2\int_{\eta_{0,\pm}}^{\eta'}\omega d\eta''} \frac{\omega'}{2\omega} d\eta'. \tag{53}$$

Note that if we choose $\eta_{0,+} = \eta_1$ ($\eta_{0,-} = \eta_2$) we can get $\xi_+(\eta) > 1$ ($\xi_-(\eta) > 1$) for some η, and a catastrophic particle creation is produced. However if we choose $\eta_{0,+} = \eta_2$ and $\eta_{0,-} = \eta_1$, then $|\xi_\pm(\eta)|$ remains small for all values of η, and consequently we have

$$\mathcal{N}_+(\eta) \cong \xi_+^2(\eta) \cong e^{-4\int_{\eta}^{\eta_2}\omega d\eta'} \left(\int_{\eta}^{\eta_2} e^{2\int_{\eta'}^{\eta_2}\omega d\eta''} \frac{\omega'}{2\omega} d\eta'\right)^2, \tag{54}$$

and

$$\mathcal{N}_-(\eta) \cong \xi_-^2(\eta) \cong e^{-4\int_{\eta_1}^{\eta}\omega d\eta'} \left(\int_{\eta_1}^{\eta} e^{2\int_{\eta_1}^{\eta'}\omega d\eta''} \frac{\omega'}{2\omega} d\eta'\right)^2. \tag{55}$$

In the next Section we apply these results to the problem of the "catastrophic particle creation in tunneling universe".

3.2 Particle Production in Tunneling Universe

Here we consider a minisuperspace model for a closed Friedmann-Robertson-Walker universe with a positive cosmological constant filled with a uniform massive scalar field conformally coupled to gravity. Working in Planck units, the Hamiltonian of the system is given by [19, 22]

$$\hat{\mathcal{H}}(a,\phi) = \frac{1}{2}\partial^2_{a^2} - V(a) + \hat{\mathcal{H}}_m(a), \tag{56}$$

with potential $V(a) \equiv \frac{a^2}{2}(1 - H^2 a^2)$, and with matter Hamiltonian

$$\hat{\mathcal{H}}_m(a) = \frac{1}{2}\left(-\partial^2_{\phi^2} + \omega^2(a)\phi^2\right) - \frac{1}{2}\omega(a), \tag{57}$$

where the frequency of the field is $\omega^2(a) \equiv m^2 a^2 + 1$.

If we consider the matter field as a small perturbation of the system, we look for solutions of the Wheeler-DeWitt equation, $\hat{\mathcal{H}}|\Phi\rangle = 0$, with the form $|\Phi(a,\phi)\rangle = \psi(a)|\Psi(a,\phi)\rangle$ [23]. Inserting this expression in the Wheeler-DeWitt equation we obtain:

$$\left[\frac{1}{2}\partial^2_{a^2}\psi - V(a)\psi\right]|\Psi\rangle + \left[\psi\frac{1}{2}\partial^2_{a^2}|\Psi\rangle + \partial_a\psi\partial_a|\Psi\rangle + \psi\hat{\mathcal{H}}_m|\Psi\rangle\right] = 0. \tag{58}$$

Assuming that ψ is the solution of the equation

$$-\frac{1}{2}\partial^2_{a^2}\psi + V(a)\psi = 0, \tag{59}$$

and making the change $\psi = e^{-iS}$, we obtain the system

$$\begin{cases} \frac{1}{2}(\partial_a S)^2 + V(a) + \frac{i}{2}\partial^2_{a^2} S = 0, \\ \frac{1}{2}\partial^2_{a^2}|\Psi\rangle - i\partial_a S\partial_a|\Psi\rangle + \hat{\mathcal{H}}_m|\Psi\rangle = 0. \end{cases} \tag{60}$$

To solve this equations we neglect the back-reaction terms, that is, the second derivative with respect to a, then we get

$$\begin{cases} \frac{1}{2}(\partial_a S)^2 + V(a) = 0, \\ -i\partial_a S\partial_a|\Psi\rangle + \hat{\mathcal{H}}_m|\Psi\rangle = 0. \end{cases} \tag{61}$$

The first equation is the classical Hamilton-Jacobi equation, and the second one is the quantum Schrödinger equation. Effectively, the potential V has two real turning points $a_1 = 0$ and $a_2 = H^{-1}$ and there is a classically allowed region and a classically forbidden one. In the classical region $a > H^{-1}$ we choose a wave traveling to the right (tunneling boundary condition), that is, we choose $S(a) = \int_{H^{-1}}^{a}\sqrt{-2V(a)}da$. In the Euclidean region $0 < a < H^{-1}$, one must choose $S(a) = -i\int_0^a\sqrt{2V(a)}da$. Finally introducing the conformal time

$$\begin{cases} \frac{da}{d\eta} \equiv \partial_a S \to \eta = \int_{H^{-1}}^{a}\frac{da}{\sqrt{-2V(a)}} & \text{when } a > H^{-1} \\ \frac{da}{d\eta} \equiv i\partial_a S \to \eta = -\int_a^{H^{-1}}\frac{da}{\sqrt{2V(a)}} & \text{when } 0 < a < H^{-1}, \end{cases} \tag{62}$$

we can see that the second equation of (61) behaves

$$\begin{cases} i\partial_\eta|\Psi\rangle = \hat{\mathcal{H}}_m|\Psi\rangle & \text{when } a > H^{-1} \leftrightarrow \eta > 0 \\ \partial_\eta|\Psi\rangle = \hat{\mathcal{H}}_m|\Psi\rangle & \text{when } 0 < a < H^{-1} \leftrightarrow -\infty < \eta < 0. \end{cases} \tag{63}$$

Then, since the wave-function of the universe is "author-dependent" [24], to avoid the catastrophic particle production during the tunneling and consequently to assure that the back-reaction can be neglected, we choose as boundary condition, that the universe nucleates via tunneling from nothing, in the matter vacuum state (defined using the diagonalization approach). Then as we have seen in Sect. 3.1, in the classically forbidden region the particle production will be very small, and in the classically allowed region we will have the usual quantum fields theory in curved space-time. In resume, at $a = H^{-1} \leftrightarrow \eta = 0$, we choose

$$|\Psi\rangle = |0\rangle \equiv \left(\frac{\omega(H^{-1})}{\pi}\right)^{1/4} e^{-\frac{\omega(H^{-1})}{2}\phi^2}. \tag{64}$$

Remark 3.1 In [22] the authors, essentially, select as a vacuum state the Gaussian state $|\Phi_+(\eta)\rangle$ defined in (48), with $\eta_{0,+} = 0$ and $\Re(\nu'_+(0)/\nu_+(0)) < 0$, with this interpretation of the vacuum state, there is not particle production and the back-reaction becomes small, this means that the approximation (61) holds, and effectively the universe undergo a tunneling transition from nothing.

4 Discussion

We have showed some differences that exist between the diagonalization method and the adiabatic vacuum prescription studying a no-trivial example: the particle production in the de Sitter space-time. We have also discussed the problem of the catastrophic particle production in tunneling universe, our prescription to solve this problem is very plausible, we assume that the universe nucleates in the matter vacuum state and then, in the classical allowed region, one obtains the standard quantum fields theory in curved space-time.

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